Thm 1.2 Let A = set of convex combination of $\vec{x}_1, \dots, \vec{x}_k$

$B = \langle \{\vec{x}_1, \dots, \vec{x}_k\} \rangle$

Then $A = B$

Pf: (i) A is convex:

$\forall \vec{y}_1, \vec{y}_2 \in A \quad \vec{y}_1 = \sum_{i=1}^{k} M_i \vec{x}_i, \quad \vec{y}_2 = \sum_{i=1}^{k} N_i \vec{x}_i \qquad \sum M_i = \sum N_i = 1, \ M_i, N_i \geq$

$\forall \lambda \in [0,1], \quad \lambda \vec{y}_1 + (1-\lambda) \vec{y}_2 = \sum_{i=1}^{k} (\lambda M_i + (1-\lambda) N_i) \vec{x}_i$

where $\lambda M_i + (1-\lambda) N_i \geq 0$ and $\sum_{i=1}^{k} (\lambda M_i + (1-\lambda) N_i) = 1$

$\therefore \quad \lambda \vec{y}_1 + (1-\lambda) \vec{y}_2 \in A.$

(ii) $B \subseteq A$: Want to show $\{\vec{x}_1, \vec{x}_2 \dots \vec{x}_k\} \subseteq A$

then $B = \langle \{\vec{x}_1, \dots \vec{x}_k\} \rangle \subseteq \langle A \rangle = A \quad (\because A \text{ convex})$

for each $\vec{x}_i$, $\quad \vec{x}_i = 1 \vec{x}_i + \sum_{\substack{j=1 \\ j \neq i}}^{k} 0 \vec{x}_j \in A.$

(iii) $A \subseteq B$: By induction

(a) $k = 1$ is true:

$A_1 = \{\vec{x}_1\} \subseteq \langle \{\vec{x}_1\} \rangle = B_1 \quad (\because$ a singleton set is convex$)$

(b) suppose $k-1$ is true:

i.e. A_{k-1} = set of convex combination of $\vec{x}_1, \dots, \vec{x}_{k-1} \subseteq \langle \{\vec{x}_1, \dots, \vec{x}_{k-1}\} \rangle$

(c) show that k is true: i.e. $\forall \vec{x} \in A_k$ show that $\vec{x} \in B_k = \langle \{\vec{x}_1, \dots, \vec{x}_k\} \rangle$

$\forall \vec{x} \in A_k$: $\quad \vec{x} = \sum_{i=1}^{k} M_i \vec{x}_i \qquad \sum_{i=1}^{k} M_i = 1, \ M_i \geq 0 \ \forall i$

Case 1: $\vec{x} = \vec{x}_k$ (i.e. $M_k = 1$) $\Rightarrow \vec{x} = \vec{x}_k \in B_k$ as B_k smallest convex set containing $\{\vec{x}_1, \dots, \vec{x}_k\}$

Case 2: $\vec{x} \neq \vec{x}_k$ (i.e. $M_k < 1$)

Case 2 ($M_K < 1$) define $\vec{y} = \sum_{i=1}^{k-1} \left(\frac{M_i}{1-M_K} \right) \vec{x}_i$

$$\frac{M_i}{1-M_K} \geq 0 \qquad \sum_{i=1}^{k-1} \frac{M_i}{1-M_K} = 1$$

$\therefore \vec{y} \in A_K$ = convex combination of $\vec{x}_1, \ldots \vec{x}_{K-1}$
$\subseteq B_K$

Since $\vec{x}_K \in B_K$,

$$\therefore \quad (1-M_K)\vec{y} + M_K x_K \in B_K$$

$$\begin{aligned} &(1-M_K) \sum_{i=1}^{k-1} \left(\frac{M_i}{1-M_K} \right) \vec{x}_i + M_K \vec{x}_K \\ &= \sum_{i=1}^{k} M_i \vec{x}_i = \vec{x} \end{aligned}$$

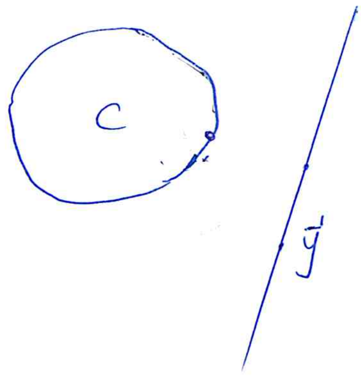
Thm 1.3 Supporting Hyperplane.

C closed & convex, $\vec{y} \notin C$,

then $\exists$ hyperplane $X = \{ \vec{x} \mid \vec{a}^T \vec{x} = z \} \ni \vec{y}$

& $C \subseteq X^+ = \{ \vec{x} \mid \vec{a}^T x \gneq z \}$

or $C \subseteq X^- = \{ \vec{x} \mid \vec{c}^T \vec{x} < z \}$

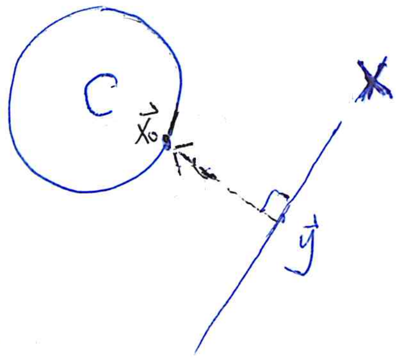


Pf: Let $\delta = \inf\limits_{\vec{x} \in C} |\vec{x} - \vec{y}| > 0$ $\quad (\because \vec{y} \notin C)$

$\because$ C is closed, inf is attained in C,

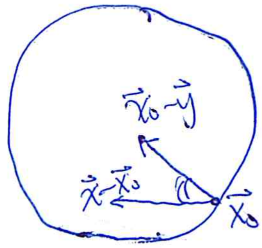
say at $\vec{x}_0 \in C$ i.e.

$$\delta = |\vec{x}_0 - \vec{y}| \le |\vec{x} - \vec{y}| \quad \forall \vec{x} \in C$$



Define $\begin{cases} \vec{a} = \vec{x}_0 - \vec{y} & \text{ \& } X = \{ \vec{x} \mid \vec{a}^T \vec{x} = z \} \\ \vec{a}^T \vec{y} = z \end{cases}$ Thus $\vec{y} \in X$

Claim $C \subseteq X^+ = \{ \vec{x} \mid \vec{a}^T \vec{x} > z \}$



$\forall \vec{x} \in C$ $\quad \lambda \vec{x} + (1-\lambda) \vec{x}_0 \in C \quad \lambda \in [0,1]$

$$\therefore \ |\lambda \vec{x} + (1-\lambda)\vec{x}_0 - \vec{y}|^2 \ge |\vec{x}_0 - \vec{y}|^2$$

$$\parallel$$

$$|\lambda(\vec{x} - \vec{x}_0) + \vec{a}|^2 \ge |\vec{a}|^2$$

$$|a|^2 + 2\lambda \vec{a}^T(\vec{x} - \vec{x}_0) + \lambda^2 |\vec{x} - \vec{x}_0|^2 \ge |a|^2$$

$$\Rightarrow \quad 2\lambda \vec{a}^T(\vec{x} - \vec{x}_0) + \lambda^2 |\vec{x} - \vec{x}_0|^2 \ge 0 \qquad \forall \lambda \in [0,1]$$

let $\lambda \to 0^+$ $\Rightarrow$ $\vec{a}^T(\vec{x} - \vec{x}_0) \ge 0$ (i.e. angle ∡ $\vec{a}$ & $(\vec{x} - \vec{x}_0) < 90°$)

$$\Rightarrow \quad \vec{a}^T \vec{x} \ge \vec{a}^T \vec{x}_0$$

$$\parallel$$

$$\vec{a}^T(\vec{a} + \vec{y}) = |a|^2 + \vec{a}^T y = |a|^2 + z = \delta^2 + z > z$$